No Arbitrage equation

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No arbitrage equation in financial markets

Bonds

Suppose we are in the bond market and we are interested in determining bond prices and yields relationships.
If we want to invest 1€today in a one-year bond, we expect to receive what we invested plus an interest rate on the investment. The same will hold for 1€invested today for 2 years: in that case our final payment will be composed of our initial investment, interest rate of first year on the investment made and interest rate of second year on the amount invested, plus what we would have gotten back after first year.
In this way we have a predetermined quantity invested today (1€) and we can compute how to get in the future accordingly. Long story short:

t	$t+1$	$t+2$
1€	$1€(1+i_t)$	$1€(1+i_t)(1+i_{t+1})$

At the same time, we could be interested in having a determined quantity of money in the future and calculate what is the corresponding value to be invested tomorrow. What we obtain is called **present discounted value**. *Present*, since it looks at the today's value of a specific amount tomorrow. *Discounted*, since it takes into account the value of money over time, more precisely: think as if, for a quantity invested in a safe bond asset, the value of today is lower than tomorrow, since you get interest payments over time.
Present value is derived as follows: define with $€P_t$[1] any amount of money you want to invest today, to get a final payment tomorrow equal to $Z_{t+1} = €P_t(1+i_t)$. What will be the value

[1] Think of it as the €1 above.

of investment today in terms of your final payment received?

$$€P_t = \frac{Z_{t+1}}{(1+i_t)}$$

Now, adding multiple periods over time and uncertainty, we get the formula for **expected discounted value**:[2]

$$€V_t^e = €Z_t + \frac{1}{(1+i_t)}€Z_{t+1} + \frac{1}{(1+i_{t+1}^e)}€Z_{t+2} + ... \tag{1}$$

Linking Bond Prices

Taken for granted the knowledge of concepts like *maturity*, *discount* and *coupon* bonds, we can introduce the bond price at present value for different maturities. As already spotted above, bond price at present value for 1-year maturity is given by:

$$€P_{1t} = \frac{Z_{t+1}}{(1+i_t)}$$

While for that of 2-years maturity:

$$€P_{2t} = \frac{Z_{t+2}}{(1+i_t)(1+i_{t+1}^e)}$$

A question rises spontaneously: is it better to invest in 1-year bond today, get the payment after 1 year and reinvest it in another 1-year bond to get the final payment after one more year,
or investing directly in a 2-years bond today and get back the final payment after 2 years from now? The answer should be, and actually is, "it's the same, otherwise arbitrage opportunities would rise up". Imposing the **no-arbitrage condition** means equating the final payment

	t	$t+1$	$t+2$
1-yr ($t \to t+1$)	$€P_{1t}$	$Z_{t+1} = €P_{1t}(1+i_t)$	–
1-yr ($t+1 \to t+2$)	–	P_{t+1}^e	$Z_{t+2} = €P_{1,t+1}^e(1+i_{t+1}^e)$
2-yrs ($t \to t+2$)	$€P_{2t}$	–	$Z_{t+2} = €P_{2t}(1+i_t)(1+i_{t+1}^e)$

Table 1: Relationship between 1-year and 2-years bonds.

Z_{t+2} coming from either 1-yr bond at $t+1$ and the 2-yrs bond. The result is given below:

$$€P_{1,t+1}^e(1+i_{t+1}^e) = €P_{2t}(1+i_t)(1+i_{t+1}^e)$$
$$€P_{1,t+1}^e = €P_{2t}(1+i_t)$$

Or, equivalently,

$$(1+i_t) = \frac{P_{1,t+1}^e}{P_{2t}} \tag{2}$$

[2]Note that equation (1) contains present payment that was not there before, and P_t is replaced with V_t to be consistent with Blanchard book.

Linking Bond Yields

In the previous section we linked prices of different bonds. Now we see how yields of those bonds relate accordingly.

Suppose there exist a bond with *n-year* interest rate, i.e. an interest rate constant over time, for instance a 2-yrs bond whose yield is defined as i_{2t}:

$$€P_{2t} = \frac{Z_{t+2}}{(1+i_{2t})^2} \tag{3}$$

Now we want to link this constant yield to the changing yield over time. We link them by equating their price P_{2t}.

From table 1,

$$€P_{2t} = \frac{Z_{t+2}}{(1+i_t)(1+i^e_{t+1})} \tag{4}$$

From equations (3), (4), we obtain:

$$\begin{aligned}
\frac{Z_{t+2}}{(1+i_2t)^2} &= \frac{Z_{t+2}}{(1+i_t)(1+i^e_{t+1})} \\
(1+i_{2t})^2 &= (1+i_t)(1+i^e_{t+1}) \\
\ln(1+i_{2t})^2 &= \ln(1+i_t) + \ln(1+i^e_{t+1}) \\
2i_{2t} &\approx i_t + i^e_{t+1} \\
i_{2t} &\approx \frac{1}{2}(i_t + i^e_{t+1})
\end{aligned}$$

Simnilarly, for a 3-yrs bond:

$$\begin{aligned}
(1+i_{3t})^3 &= (1+i_t)(1+i^e_{1,t+1})(1+i^e_{1,t+2}) \\
\ln(1+i_{3t})^3 &= \ln(1+i_t) + \ln(1+i^e_{1,t+1}) + \ln(1+i^e_{1,t+2}) \\
3i_{3t} &\approx i_t + i^e_{t+1} + i^e_{1,t+2} \\
i_{3t} &\approx \frac{1}{3}(i_t + i^e_{t+1} + i^e_{1,t+2})
\end{aligned}$$

Investing in a multiple year bond is more risky, since the price at which it could be sold is not known. Adding **risk-premium (x)** to the multiple-year bond, we obtain a similar expression as above, which is also true for bonds with a longer maturity than just 2-years:

$$i_{2t} \approx \frac{1}{2}(i_t + x + i^e_{t+1}) \tag{5}$$

Finally, following the same logic as above, the bond market no-arbitrage general formula is given by:

$$i_{nt} \approx \frac{1}{n}(i_{1t} + x_{nt} + i^e_{1,t+1} + i^e_{1,t+2} + ... + i^e_{1,t+n-1}) \tag{6}$$

The equation proves that the long-term interest rates reflects current and future expected short-term interest rates and risk premium.